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The replica-symmetry-breaking solution and the 'SK limit' of the Hopfield model

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Abstract. The replica-symmetry-breaking solution of the Hopfield model is formulated in order to investigate the multivalley structure of the rugged free energy. It is found that the resulting variational equations are equivalent to those for the Sherrington–Kirkpatrick (SK) model as a limit. Numerical solutions are obtained for the spin-glass phase. These provide us with a weight distribution of the valleys.

1. Introduction

Recently the words 'spin glasses' (SG) have been seen in diverse fields [1,2]. They have also appeared in many papers for neural network models. If we consider the Hopfield model [3,4] as a globally and not uniformly connected spin system, various SG-like properties emerge. Its most significant characteristics are the existence of a large number of metastable states and multivalley structures in phase space which are also typical examples of 'broken ergodicity' [5]. These properties were originally derived from an interpretation of the replica-symmetry-breaking (RSB) solution given by Parisi [6] for the Sherrington and Kirkpatrick (SK) model [7] of SG, and were thought to be rather specific to the SK model. More recent studies, however, have shown that this is not the case and that they are shared by other infinite-range SG models[8]. In the present work, we will show that the Hopfield model, too, can be regarded as such a model.

Very useful for obtaining some insight into the complexity of a valley structure is the basin of attraction which gives a quantitative measure for the 'spread' of the valley. Therefore, in a previous paper, we studied numerically relationships between remanent overlaps and initial overlaps in the Hopfield model using Monte Carlo simulations and finite size scaling at zero temperature ($T = 0$) [9]. This provides an intuitive understanding for the valley structure in *overlap space*. In this paper, we consider the valleys in *spin configuration space* by obtaining the full RSB solution of the Hopfield model. It turns out that the formulation obtained is equivalent to the one for the original SK model in the limit $\sqrt{\alpha} \to \infty$, where α ($\equiv p/N$) is the rate of memory-loading, p the number of random patterns for memories and N the system size. In this way we also proved conjectures by several authors [4, 10, 11] regarding the SG limit of the Hopfield model. Furthermore the full RSB solutions are obtained numerically for the SG phase.

The replica-symmetric (RS) solutions for the Hopfield model were investigated fully by Amit, Gutfreund and Sompolinsky (AGS) [4]. One of their main results is that the RS

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solutions are stable in almost all regions that belong to the 'retrieval phase' where the system behaves as an associative memory. They derived the mean-field equations and critical storage capacity $\alpha_c \sim 0.137$ at $T = 0$ by extrapolating the RS solutions to the RSB region. The RS solutions for the SG states, however, are unstable in the entire region $T < T_g$ (the SG transition temperature) and those for the retrieval states become unstable at $T < T_R$, the generalized De Almeida–Thouless line [4, 12] for the Hopfield model. To overcome this difficulty, Crisanti *et al* have concentrated on estimating the critical storage capacity by considering the first-step RSB scheme (which effectively means taking $K = 1$ in the appendix) and they obtained the value $\alpha_c \sim 0.145$ [13]. On the other hand, in this paper, the full RSB scheme will be employed to extract several pieces of information regarding the multivalley structure of the free energy of the Hopfield model, which interpolates between the Mattis model and the SK model. Furthermore, we will also reveal non-trivial asymptotic shifts of the system to the SK model, i.e. the 'SK limit', using the notion of the rugged free energy landscape.

In section 2, the weights of the valleys will be defined first. Next, the statistical mechanical interpretation of the RSB solution and the relationships between the order parameter function and the weight distribution of valleys will be given. In section 3, we will briefly review the Hopfield model and apply the full RSB scheme. It turns out that the formulation obtained is equivalent to that for the original SK model as a limit. We solved numerically the resulting variational equations for the SG phase with rather large α's and some values of temperature. These solutions provide us with the weight distribution of the valleys. Our results are summarized in section 4.

2. Valleys

First let us define the weights of the valleys. When the phase space is divided into a number of valleys indexed by k, the weight of kth valley, W_k, is defined as the probability with which a randomly chosen initial state is in the kth valley. If the system is at equilibrium, W_k can be represented as

$$W_k = \exp(-\beta f_k) / \sum_j \exp(-\beta f_j) \qquad \sum_k W_k = 1 \tag{1}$$

where f_k refers to the free energy of kth valley and $\beta = 1/T$. In connection with W_k, Mézard *et al* [14] have defined a quantity

$$y = \left\langle \sum_k W_k^2 \right\rangle_J \tag{2}$$

where $\langle \cdots \rangle_J$ denotes the sample average. The quantity y roughly shows the number and the width of the valleys and gives the relative distribution of the weights of the valleys. For example, there is only one large valley for $y = 1$, while there are many valleys with small weights for $y \simeq 0$.

Secondly, let $m_{i,\rho} \equiv \langle S_i \rangle_\rho$ be the magnetic moment of the ith spin of the ρth pure state. Overlaps of magnetization between two pure states are defined as follows:

$$q_{\rho\sigma} \equiv \frac{1}{N} \sum_{i=1}^{N} m_{i,\rho} m_{i,\sigma}. \tag{3}$$

It is noted that the self-overlap, $q_{\rho\rho}$ ($\equiv q^{EA}$), denotes the Edwards–Anderson order parameter. Using Boltzmann weights P_ρ, P_σ, the distribution of the overlap q is represented as

$$P(q) \equiv \left\langle \sum_{\rho,\sigma} P_\rho P_\sigma \delta(q - q_{\rho\sigma}) \right\rangle . \tag{4}$$

The cumulative distribution function is defined as

$$Y(q) \equiv \int_q^1 \mathrm{d}q' P(q') . \tag{5}$$

$Y(q)$ also gives the density of pure states with an overlap larger than q. If we now consider the function

$$x(q) \equiv 1 - Y(q) = \int_{-1}^{q} \mathrm{d}q' P(q') \tag{6}$$

we see that its inverse, $q(x)$, coincides exactly with the RSB solution of the SG order parameter, which is why it is called the physical interpretation of the RSB solution [6, 15]. The case $Y(q^{EA})$ refers to the self-overlap of a pure state by definition and is equivalent to y in (2) since $W_k = P_\rho$ (one pure state for one valley). Furthermore, $Y(q^{EA})$ is given by the length of the plateau of $q(x)$ whose functional form depends on α. Hence, by obtaining the RSB solutions for various values of α, the variation of y can be estimated.

3. The RSB scheme for the Hopfield model

3.1. The AGS theory

The Hamiltonian for the Hopfield model is given by

$$H = -\tfrac{1}{2} \sum_{i,j}^{N} J_{ij} S_i S_j - \sum_{\nu}^{s} h^\nu \sum_i \xi_i^\nu S_i \qquad (S_i = \pm 1) \tag{7}$$

where h^ν is a field conjugate to one of a finite number ($s \ll p$) of 'condensed patterns' $\{\xi_i^\nu\}$. J_{ij} is an interaction constructed from p random patterns ($\xi_i^\mu = \pm 1$, $\mu = 1, \ldots, p$) as

$$J_{ij} \equiv \frac{1}{N} \sum_{\mu=1}^{p} \xi_i^\mu \xi_j^\mu \qquad J_{ii} = 0 . \tag{8}$$

AGS have introduced the free energy of the Hamiltonian (7) as follows:

$$f_n = \frac{\alpha}{2} + \frac{1}{2n} \sum_{\mu=1}^{s} \sum_{a=1}^{n} (m_a^\mu)^2 + \frac{\alpha}{2\beta n} \mathrm{Tr} \ln((1-\beta)\mathbf{I} - \beta\mathbf{Q}) + \frac{\alpha\beta}{2n} \sum_{a\neq b}^{n} r_{ab} q_{ab} - \frac{1}{n\beta} \langle \ln Z_0 \rangle$$
$$\text{as} \quad n \to 0 \tag{9}$$

where

$$Z_0 = \mathrm{Tr}_{\{n\}} \exp\left(\tfrac{1}{2} \sum_{a\neq b} r_{ab} \sigma_a \sigma_b + \sum_a y_a \sigma_a \right) \tag{10}$$

$$\sigma_a = \sqrt{\alpha}\beta S^a \qquad y_a = \sum_\nu (m_a^\nu + h^\nu)\xi^\nu / \sqrt{\alpha} . \tag{11}$$

Here **I** is a unit matrix with $n \times n$ elements, while **Q** is a replica matrix which gives the order parameter q_{ab}. By $\mathrm{Tr}_{\{n\}}$ we denote explicitly that the trace is taken over n-replicated binomial spins. Definitions for order parameters are given as follows:

$$m_a^\nu = \left\langle\!\!\left\langle \frac{1}{N}\sum_i \xi_i^\nu \langle S_i^a \rangle \right\rangle\!\!\right\rangle \qquad \nu = 1 \ldots s \tag{12}$$

$$q_{ab} = \left\langle\!\!\left\langle \frac{1}{N}\sum_i \langle S_i^a \rangle \langle S_i^b \rangle \right\rangle\!\!\right\rangle \tag{13}$$

$$r_{ab} = \frac{1}{\alpha}\sum_{\mu>s}^{p} \langle\!\langle m_a^\mu m_b^\mu \rangle\!\rangle \tag{14}$$

where

$$\langle\!\langle \mathrm{O}\{\xi^\nu\} \rangle\!\rangle \equiv \frac{1}{2^s}\sum_{\nu=1}^{s}\sum_{\xi^\nu=\pm 1} \mathrm{O}\{\xi^\nu\} \tag{15}$$

denotes the random average for s 'condensed patterns'. In addition to q_{ab} in the SK model, there appears one more order parameter, r_{ab}, depending on two replica indices. AGS have shown that the RS solution ($q_{ab} = q$, $r_{ab} = r$) is stable in almost all regions which belong to the so-called 'retrieval phase', and by extrapolating to the RSB region they derived mean-field equations at zero temperature ($\beta \to \infty$). However, in the limit $\beta \to \infty$ the full RSB should be considered. Furthermore, the RS solutions for the SG states are unstable in all regions for the SG phase. In the next section we will consider Parisi's RSB scheme for arbitrary values of α to extract the valleys' variation to α. In particular, the RSB discussion is important for understanding the equivalence between the SK model and the Hopfield model with large α.

3.2. The full RSB scheme

Following the formalism of AGS and Parisi's recipe of taking a continuum limit of partitions of replica matrices (*Parisi gauge*), the free energy for the Hopfield model can be represented as the functional

$$f_p = \frac{\alpha\beta}{2}\left(r(1) - \int_0^1 r(x)q(x)\,\mathrm{d}x\right) - \int_{-\infty}^{\infty}\frac{\mathrm{d}z}{\sqrt{2\pi r(0)}}\exp\left(\frac{-z^2}{2r(0)}\right)\langle\!\langle g(0, h+z)\rangle\!\rangle$$
$$+\frac{\alpha}{2} + \frac{1}{2}\sum_\nu^s (m^\nu)^2 + \frac{\alpha}{2\beta}\left\{\frac{-\beta q(0)}{1-\chi(0)} + \ln[1-\chi(1)] + \int_0^1 \frac{\mathrm{d}x}{x}\frac{\dot{\chi}}{1-\chi(x)}\right\} \tag{16}$$

which is maximized by the order parameter functions $q(x)$ and $r(x)$, and minimized by m^ν. Here $q(x)$ and $r(x)$ correspond to the order parameters (13), (14) in the continuum limit, i.e. the SG order parameter function and the order parameter function describing the noise due to the *uncondensed* patterns, respectively. m^ν denotes the average overlap of the states and the νth memorized pattern (12). The terms including $\chi(x)$ correspond to the third term in (9) and they are obtained by taking the continuum limits of the eigenvalues of the replica matrix **Q** in (9) since it has a recursive structure. The details of the limiting process are

left to the appendix. Furthermore, $g(x,z)$ $(0 \leqslant x \leqslant 1,\ -\infty < z < \infty)$ is a solution of the following partial differential equation:

$$\dot{g} = -\frac{\dot{r}}{2}\left(g'' + \beta x g'^2\right) \qquad \text{(Parisi equation)} \qquad \left(\dot{A} \equiv \frac{\partial A}{\partial x} \qquad A' \equiv \frac{\partial A}{\partial z}\right) \tag{17}$$

$$g(1,z) = \beta^{-1} \ln 2\cosh\sqrt{\alpha}\beta z\,. \tag{18}$$

The derivations of the terms containing $g(x,z)$ are similar to the case for the SK model [6, 8, 16]. $\chi(x)$ and h correspond to the local susceptibilities at scale x and the effective field, respectively (h^ν denotes the external field, conjugate with νth condensed pattern), as

$$\chi(x) = \beta\left(1 - xq(x) - \int_x^1 q(\tilde{x})\,\mathrm{d}\tilde{x}\right) \tag{19}$$

$$h = \frac{1}{\sqrt{\alpha}}\sum_{\nu=1}^{s}(m^\nu + h^\nu)\xi^\nu. \tag{20}$$

Equation (16) cannot directly be maximized numerically since $g(0,z)$ depends implicitly on $r(x)$ through (17). Therefore we apply the schemes by Sommers *et al* [17] and Nemoto [18] to obtain the order parameter functions $q(x), r(x)$. They make $g(x,z)$ independent of $r(x)$ by introducing a Lagrange multiplier function $P(x,z)$. The new functional to be maximized is then defined as

$$f_s[r,q,g,P] = f_p[r,q,g] + \int_{-\infty}^{\infty}\mathrm{d}z\,P(1,z)\left\{g(1,z) - \frac{1}{\beta}\ln 2\cosh\sqrt{\alpha}\beta z\right\}$$
$$-\int_0^1 \mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}z\,P(x,z)\left\{\dot{g} + \frac{\dot{r}}{2}\left(g'' + \beta x g'^2\right)\right\}. \tag{21}$$

The equations to be solved can be obtained by taking functional derivatives of (21) with respect to q, r, g, P and m^ν:

$$\dot{M} = -\frac{\dot{r}}{2}(M'' + 2\tilde{\beta}xMM') \qquad \left(M \equiv \frac{g'}{\sqrt{\alpha}}\right) \tag{22}$$

$$M(1,z) = \tanh\tilde{\beta}z \tag{23}$$

$$\dot{P} = \frac{\dot{r}}{2}(P'' - 2\tilde{\beta}x(PM)') \tag{24}$$

$$P(0,z) = \left\langle\!\!\left\langle \frac{1}{\sqrt{2\pi r(0)}}\exp\left(-\frac{(z-h)^2}{2r(0)}\right)\right\rangle\!\!\right\rangle \tag{25}$$

$$q(x) = \int_{-\infty}^{\infty}\mathrm{d}z\,P(x,z)M(x,z)^2 \tag{26}$$

$$\chi(x) = \frac{\tilde{\beta}}{\sqrt{\alpha}}\left(1 - xq(x) - \int_x^1 q(\tilde{x})\,\mathrm{d}\tilde{x}\right) \tag{27}$$

$$r(x) = \frac{q(0)}{(1-\chi(0))^2} + \int_0^x \mathrm{d}\tilde{x}\frac{\dot{q}(\tilde{x})}{(1-\chi(\tilde{x}))^2} \tag{28}$$

$$m^\nu = \int_{-\infty}^{\infty}\mathrm{d}z M(0,z)\left\langle\!\!\left\langle \xi^\nu \frac{1}{\sqrt{2\pi r(0)}}\exp\left(-\frac{(z-h)^2}{2r(0)}\right)\right\rangle\!\!\right\rangle \tag{29}$$

$$\tilde{\beta} \equiv \sqrt{\alpha}\beta\,. \tag{30}$$

In the above formalism $P(1,z)$ and $M(0,z)$ give the internal field distribution and the local magnetization, respectively. Furthermore, if we consider only one condensed pattern ($s=1$) and $h^1=0$, we can estimate the sample average $\langle\cdots\rangle$ in (25), (29), and obtain the following equations:

$$P(0,z)=\frac{1}{\sqrt{2\pi r(0)}}\exp\left(-\frac{(z-m/\sqrt{\alpha})^2}{2r(0)}\right) \tag{31}$$

$$m=\int_{-\infty}^{\infty}\mathrm{d}z\, M(0,z)P(0,z) \tag{32}$$

where we have written m^1 as m. It is noted that (31) and (32) coincide with the corresponding equations for the SK model in a magnetic field. Consequently, in the same way as for the SK model, the field distribution can be corrected to a non-trivial function $P(x,z)$ from the Gaussian distribution by the RS solution.

Equations (22)–(26) are similar to those for the SK model in which β is scaled by $\sqrt{\alpha}$ while (27) and (28) are specific to the present analysis. If we now consider the limit $\sqrt{\alpha}\to\infty$ while keeping $\tilde{\beta}$ constant, we obtain that $\chi(x)\to 0$, $r(x)=q(x)$ for arbitrary x and that $m=0$. We thus find that equations (22)–(32) formally coincide with those of the SK model and obtain the first result of this paper, i.e. we establish the 'SK limit' of the Hopfield model. In this way we furthermore prove the correctness of similar conjectures for the 'SG limit' of the Hopfield model [4, 10, 11].

Here it is noted that the conditions $\dot{q}=\dot{r}=0$ again yield the mean-field equations obtained from the RS discussion. Furthermore, the differentiation of (26) with respect to x gives the condition

$$\frac{1}{[1-\chi(x)]^2}\int_{-\infty}^{\infty}\mathrm{d}z\, P(x,z)M'(x,z)^2=1 \tag{33}$$

or

$$\dot{q}=0. \tag{34}$$

This equation is equivalent to the condition for marginal stability in replica space. If we apply the RS solutions to (33), one can easily obtain the equation for the generalized Almeida–Thouless line of the Hopfield model [4] as

$$\alpha\beta^2\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{d}z\,\mathrm{e}^{(-z^2/2)}\mathrm{sech}^4\left[\beta\left(\sqrt{\alpha r}z+(m+h^1)\right)\right]=\left[1-\beta(1-q)\right]^2 \tag{35}$$

where q and r correspond to the RS solutions of $q(x)$ and $r(x)$, respectively. h^1 denotes the external field, conjugate to the first condensed pattern.

3.3. Numerical analysis

We have solved (22)–(32) numerically for some parameters. Analytical results are known only near T_g, the SG transition temperature. Since we can interpet the linear terms of (22), (24) as diffusion equations, by introducing the Green function, the nonlinear partial differential equations (22), (25) can be transformed to the following integral equations:

$$G(x,z;x',z')=\frac{1}{\sqrt{2\pi(r(x')-r(x))}}\exp\left(-\frac{(z-z')^2}{2(r(x')-r(x))}\right) \tag{36}$$

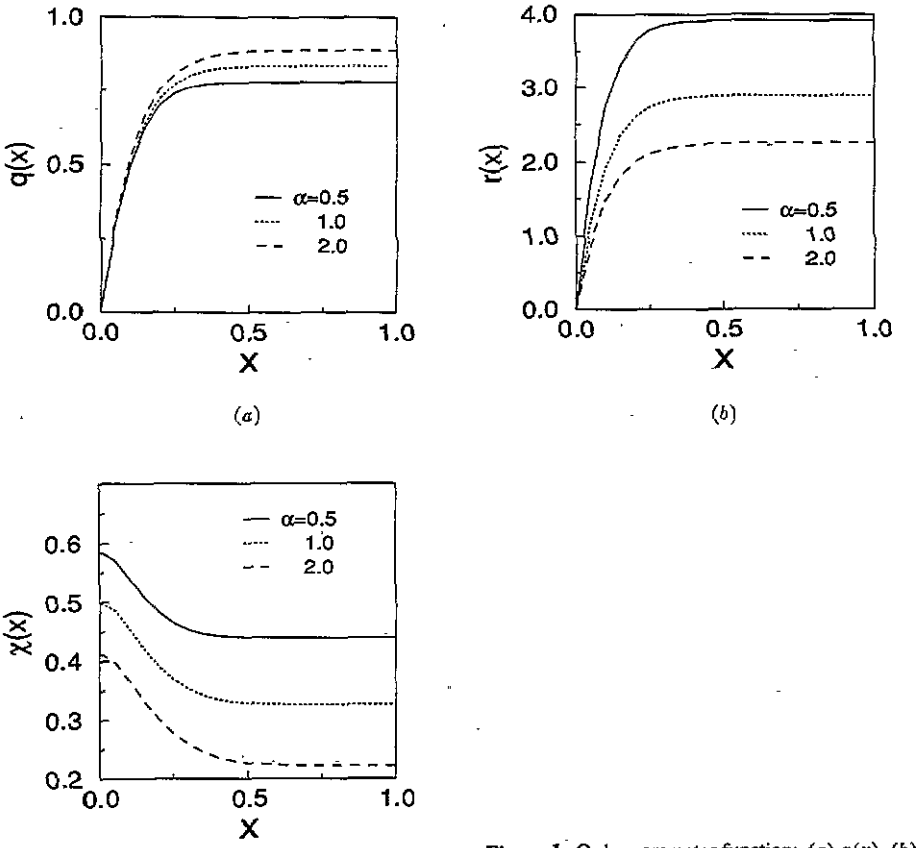

Figure 1. Order parameter function: (*a*) $q(x)$, (*b*) $r(x)$ and (*c*) $\chi(x)$ at $T = 0.5$ for $\alpha = 0.5$, 1.0 and 2.0.

$$M(x,z) = \int_{-\infty}^{\infty} dz'\, G(x,z;1,z') \tanh \tilde{\beta} z' + \int_{x}^{1} dx'\, \tilde{\beta} \dot{r}(x') x' \times \int_{-\infty}^{\infty} dz' G(x,z;x',z') M(x',z') M'(x',z') \tag{37}$$

$$P(x,z) = \frac{1}{\sqrt{2\pi r(x)}} \exp\left(-\frac{(y - m/\sqrt{\alpha})^{1/2}}{2r(x)}\right) - \int_{0}^{x} dx'\, \tilde{\beta} \dot{r}(x') x' \times \int_{-\infty}^{\infty} dz'\, G(x',z';x,z) \left(M(x',z') P(x',z')\right)' . \tag{38}$$

This enables us to obtain M, P, m, r, q and χ by means of an iterative procedure in the order (37)→(31)→(32)→(38)→(26)→(27)→(28)→(37)$\cdots$. For numerical integration and differentiation we discretized the variables x and z, dividing the intervals [0, 1] and [−7.5, 7.5] into 20 and 100 pieces, respectively, using a cubic spline for interpolation. We iterated the above procedure until the maximum variance of all the variables P, M, r, q and χ became less than 10^{-6}, which occurred typically in 700–900 iterations (dependent on α and β).

Some typical order parameter functions are shown in figure 1 for a fixed value of the temperature (T (= $1/\beta$) = 0.5) and for $\alpha = 0.5$, 1.0 and 2.0 (i.e. in the SG phase). We see that the larger α, the smaller $r(x)$ and $\chi(x)$, but the larger $q(x)$. Consequently in

the SK limit $q(x)$ and $r(x)$ are naively expected to be identical for arbitrary values of x. The internal field distribution $P(1,z)$ is represented in figure 2 for several α's and T's. It is expected that the smaller α, the nearer the system approaches the RS region at the temperature considered. Then we can see that a Gaussian form will be recovered for smaller α or higher T, while a double-peaked form of $P(1,z)$ emerges for larger α or lower T. Figure 3 concerns the variation of y for α at $T = 0.5$. We estimate the values of y, the length of the plateau, by the point where $dq(x)/dx$ is less than 10^{-3}. We see that y can be scaled as $y \sim \alpha^{-\gamma}$ ($\gamma = 0.5 \pm 0.02$). Consequently, as α gets smaller the valleys with a large weight increase their relative frequency proportionally. Moreover, the larger α, the more the number of relatively small basins increases. Here we would like to stress that an exponent of $\gamma \approx 0.5$ implies that for large α, y scales according to the SK limit ($\gamma = 0.5$) and not according to the SG limit ($\gamma = 1.0$) as thus far thought. Finally, the α dependence of $\chi(0)$ is given as circles in figure 4, which shows good agreement with the line $\chi(0) = 1/(1+\sqrt{\alpha})$ obtained analytically in [4]. It also indicates that the accuracy of our numerical analysis was sufficient.

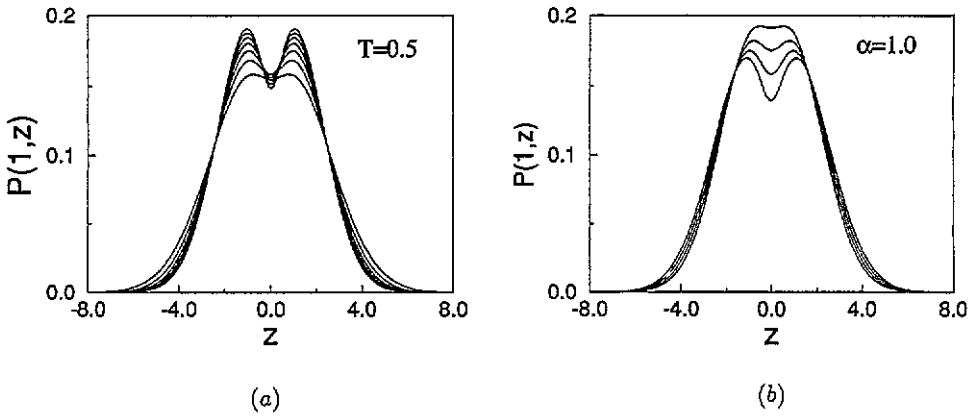

Figure 2. The internal field distribution $P(1,z)$, (*a*) at $T = 0.5$ for $\alpha = 0.5, 0.75, 1.0, 1.25, 1.5, 1.75$ and 2.0 (bottom to top) and (*b*) at $T = 0.4, 0.5, 0.6$ and 0.7 (bottom to top) for $\alpha = 1$.

4. Summary

We have formulated the RSB solution for the Hopfield model, obtained the variational equations for the order parameter functions and found that they are identical with equations for the SK model as a limit (*the SK limit of the Hopfield model*). The order parameter functions are also estimated numerically for the SG phase and from their functional form the asymptotic dependence of the parameter y (characterizing the valley structure) on α is obtained.

Let us conclude by noting that recently the author has found a gauge invariance for the free energy of the Hopfield model and formalized a free energy with the so-called *Sompolinsky gauge* [19,20]. Using the new variational equations of the formalism, we expect to obtain important results for the retrieval phase including an estimate for the critical storage capacity within the full RSB theory. The results will be reported in a forthcoming paper.

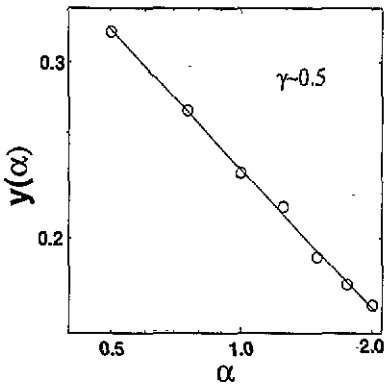

Figure 3. The asymptotic dependence of y on α at $T = 0.5$ (log–log plot).

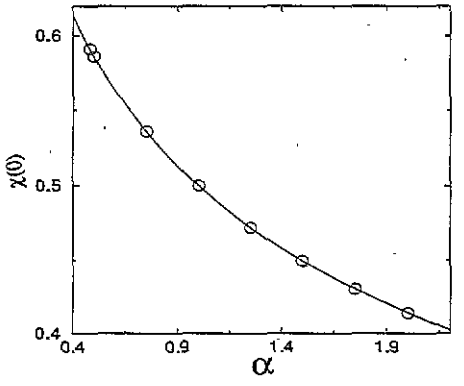

Figure 4. $\chi(0)$ (○) versus α for $T = 0.5$. The curve represents $\chi(0) = 1/T_g = 1/(1+\sqrt{\alpha})$.

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Appendix

We summarize some of the details involved in taking the $n \to 0$ limit of the term $L \equiv (\alpha/(2\beta n))\mathrm{Tr}\ln[(1-\beta)\mathbf{I} - \beta\mathbf{Q}]$ in the free energy (9). Here we note that the essence of the following calculations is the diagonalization of Parisi's replica matrix and properly taking the limits. In general, an n-dimensional Parisi matrix $\mathbf{A} = \mathbf{A}_0$ of level K is recursively defined as the Kth iterate of

$$\mathbf{A}_k = \begin{pmatrix} \mathbf{A}_{k+1} & a_k\mathbf{U}_{k+1} & \cdots & a_k\mathbf{U}_{k+1} \\ a_k\mathbf{U}_{k+1} & \mathbf{A}_{k+1} & \cdots & a_k\mathbf{U}_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_k\mathbf{U}_{k+1} & a_k\mathbf{U}_{k+1} & \cdots & \mathbf{A}_{k+1} \end{pmatrix} \tag{A1}$$

with $\mathbf{A}_{K+1} = \tilde{a}$ representing the value of the diagonal elements. $\mathbf{U}_k$ denotes the $p_k \times p_k$ matrix whose elements are all 1. Each matrix $\mathbf{A}_k$ is specified by its dimension p_k and the coefficient a_k of non-diagonal submatrices $\mathbf{U}_{k+1}$. Thus $\mathbf{A}_0$ is determined by the series of integers $n = p_0 > p_1 > \cdots > p_K > p_{K+1} = 1$, where naturally p_k needs to divide p_{k-1} in order for successive submatrices to fit correctly, and the series of coefficients $\{a_0, a_1, \ldots, a_K, a_{K+1} = \tilde{a}\}$.

By the recursive definition, the eigenvalues λ of $\mathbf{A}$ are obtained [22] as

$$\lambda_{p_0} = \sum_{i=0}^{K}(p_i - p_{i+1})a_i + \tilde{a} \tag{A2}$$

which is non-degenerate, and as

$$\lambda_k = \sum_{i=k}^{K}(p_i - p_{i+1})a_i - p_k a_k + \tilde{a} \tag{A3}$$

which are $p_0(1/p_{k+1} - 1/p_k)$-fold degenerate for each k ($k = 0, 1, \ldots, K$). The matrix $\mathbf{Q}$ in L is a Parisi matrix whose diagonal elements are zero ($\tilde{q} = 0$). Consequently, we use the eigenvalues for evaluating L and obtain

$$L = \frac{\alpha}{2\beta n}\ln\left[1 - \beta\left(1 + \sum_{i=0}^{K}(p_i - p_{i+1})q_i\right)\right] + \frac{\alpha}{2\beta}\sum_{k=0}^{K}\left(\frac{1}{p_{k+1}} - \frac{1}{p_k}\right)\ln\left[1 - \beta\left(1 + \sum_{i=k}^{K}(p_i - p_{i+1})q_i - p_k q_k\right)\right]. \tag{A4}$$

Let us write the first and the second term as $\alpha/(2\beta)L_1, \alpha/(2\beta)L_2$, respectively. If we change the order of the summation of k (like in a partial integration), L_2 can be rewritten as

$$L_2 = -\frac{1}{p_0}\ln\left[1 - \beta\left(1 + \sum_{i=0}^{K}(p_i - p_{i+1})q_i - p_0 q_0\right)\right] + \sum_{k=0}^{K-1}\frac{1}{p_k}\left\{\ln\left[1 - \beta\left(1 + \sum_{i=k-1}^{K}(p_i - p_{i+1})q_i - p_{k-1}q_{k-1}\right)\right] - \ln\left[1 - \beta\left(1 + \sum_{i=k}^{K}(p_i - p_{i+1})q_i - p_k q_k\right)\right]\right\} + \ln[1 - \beta(1 - q_K)]. \tag{A5}$$

Now, while denoting the terms in L_2 by L_{21}, L_{22}, L_{23}, we obtain after taking the continuum limit $K \to \infty$ and the limit $p_0 (= n) \to 0$,

$$L_{21} = -\frac{1}{n}\ln\left[1 - \beta\left(1 + \sum_{i=0}^{K}(p_i - p_{i+1})q_i\right)\right] - \frac{\beta q(0)}{1 - \beta(1 - \int_0^1 \mathrm{d}x\, q(x))}$$

$$L_{22} = \sum_{k=0}^{K-1}\left(\frac{1}{p_k}\right)\ln\left[1 + \frac{\beta p_k(q_{k-1} - q_k)}{1 - \beta\left(1 + \sum_{i=k}^{K}(p_i - p_{i+1})q_i - p_k q_k\right)}\right] \simeq \sum_{k=0}^{K-1}\left(\frac{1}{p_k}\right)\frac{\beta p_k(q_{k-1} - q_k)}{1 - \beta\left(1 + \sum_{i=k}^{K}(p_i - p_{i+1})q_i - p_k q_k\right)} = -\int_0^1 \mathrm{d}x \frac{\beta\dot{q}(x)}{1 - \beta\left[1 - \int_x^1 \mathrm{d}\tilde{x}\, q(\tilde{x}) - x q(x)\right]} \tag{A6}$$

$$L_{23} = \ln[1 - \beta(1 - q(1))] \tag{A7}$$

where we set $q_0 = q(0)$, $q_K = q(1)$, $p_i - p_{i+1} = \mathrm{d}x$ and $q_{j-1} - q_j = \dot{q}(x)\,\mathrm{d}x$. The first term of L_{21} is cancelled out by L_1, yielding L'_{21}. Finally, L can be represented as $\alpha/2\beta\,(L'_{21} + L_{22} + L_{23})$.

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